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THE COMPLETELY INTEGRABILITY OF A FINITE-DIMENSIONAL SYSTEM RELATED TO THE COUPLED NONLINEAR SCHRODINGER EQUATION

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ABSTRACT

The finite-dimensional complex Hamiltonian system related to the related to the complex 3 x 3 spectral problemand the coupled non-linear Schrödinger equation in the paper is obtained. With the help of the confocal involutive system, we give the proof of the completely integrability of the Hamiltonian system in the Liouville sense. Furthermore, the representation of the solution to the evolution equations is generated by the commutable flows of the finite-dimensional completely integrable system.

Keywods:- *SPECTRAL PROBLEM, EVOLUTION EQUATION, INTEGRABLE SYSTEM.*

I. INTRODUCTION

So far, higher order matrix spectral problem and complex spectral problem are all attractive to the mathematical and physical science. However, due to theoretical difficulty and the complexity of computation, the relevant reserch is relatively rare for the present. The technique of the nonlinearization [1]-[3] of Lax pairs of these spectral problems has been a powerful tool for the finding of integrable systems in the last two decades or so. How ever, the proof of the completely integrability of the Hamiltonian system in the Liouville sense is a challenging work.

we present a 3×3 AKNS matrix spectral problem[4]

$$\phi_x = M(\mathrm{u},\xi)\phi, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad M(\mathrm{u},\xi) = \begin{pmatrix} i\xi & u_1 & u_2 \\ v_1 & -i\xi & 0 \\ v_2 & 0 & -i\xi \end{pmatrix} \qquad \text{* MERGEFORMAT (1.1)}$$

where the potential $u = (u_1, v_1, u_2, v_2)^T$, $u_1 = u_1(x,t), u_2 = u_2(x,t), v_1 = v_1(x,t), v_2 = v_2(x,t)$ are complex-valued potential functions, ξ is a complex spectral parameter, $i = \sqrt{-1}$. The relation betweem this 3rd-order complex spectral problem and the associated completely integrable system is considered. We derived the related evolution equation hierarchy, one of which is often referred to on the literature as the coupled nonlinear Schrödinger equation

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} -iu_{1_{xx}} + 2iu_1(|u_1|^2 + |u_2|^2) \\ -iu_{2_{xx}} + 2iu_2(|u_1|^2 + |u_2|^2) \end{pmatrix}, \qquad \text{* MERGEFORMAT (1.2)}$$

which is used by Manakov for studying the propagation of the electric field in a waveguide[5]. Each equation governs the evolution of one of the components of the field transverse to the direction of propagation. Also it can be derived as a model for wave propagation under conditions similar to those where nonlinear Schrödinger equation applies and there are two wavetrains moving with nearly the same group velocity[6]. In recent years, this system is widely studied[6,7]and used as a key model in the field of optical solitons in fibers[7,8]to explain how the solitons waves transmit in optical fiber, what happens when the interaction among optical solitons influences directly the capacity and quality of communication and so on[9-11].

II. THE EVOLUTION EQUATIONS AND THEIR LAX PAIRS EQUATION SECTION (NEXT)

By solving the stationary zero-curvature equation:

$$V_x - [M, V] = 0, \qquad \text{* MERGEFORMAT (2.1)}$$

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Where

$$V=(\mathrm{V}_{lk})_{3\times 3},\qquad V_{lk}=\sum_{j=0}^{\infty} a_{lk}^{(j-1)}\xi^{-j}. \qquad \text{* MERGEFORMAT (2.2)}$$

We have the isospectral evolution equations:

$$u_{t_m}=Jg_m=Kg_{m-1},\ \ \mathrm{m}=1,2,3,\ldots \qquad \text{* MERGEFORMAT (2.3)}$$

By (2.1)-(2.3), we have

Theorem 2.1.

$$\begin{cases}\phi_x=M(u,\xi)\phi\\ \phi_{t_m}=V_m(u,\xi)\phi,\end{cases} \qquad \text{* MERGEFORMAT (2.4)}$$

is the Lax forms of evolution equations (2.3). In other words, the hierarchy of solition equations(2.3) is a isospectral compatible condition of (2.4).

Especially, if we take

$$a_{11}^{(0)}=a_{22}^{(0)}=a_{33}^{(0)}=a_{23}^{(0)}=a_{32}^{(0)}=0, \qquad \text{* MERGEFORMAT (2.5)}$$

$$g_0=(a_{21}^{(0)},a_{12}^{(0)},a_{31}^{(0)},a_{13}^{(0)})^T=(2v_1,2u_1,2v_2,2u_2)^T,\ \text{* MERGEFORMAT (2.6)}$$

$$g_1=(a_{21}^{(1)},a_{12}^{(1)},a_{31}^{(1)},a_{13}^{(1)})^T=(iv_{1x},-iu_{1x},iv_{2x},-iu_{2x})^T,\ \text{* MERGEFORMAT (2.7)}$$

By (2.3),

$$u_{t_1}=\begin{pmatrix}u_1\\ v_1\\ u_2\\ v_2\end{pmatrix}_{t_1}=\begin{pmatrix}2u_{1_x}\\ 2v_{1_x}\\ 2u_{2_x}\\ 2v_{2_x}\end{pmatrix}\qquad (a\ \ trivial\ \ case).$$

$$u_{t_2}=\begin{pmatrix}u_1\\ v_1\\ u_2\\ v_2\end{pmatrix}_{t_2}=\begin{pmatrix}-iu_{1_{xx}}+2iu_1(u_1v_1+u_2v_2)\\ iv_{1_{xx}}-2iv_1(u_1v_1+u_2v_2)\\ -iu_{2_{xx}}+2iu_2(u_1v_1+u_2v_2)\\ iv_{2_{xx}}-2iv_2(u_1v_1+u_2v_2)\end{pmatrix},$$

if $u_1=v_1^*$, it is exactly the coupled nonlinear Schrödinger equation (1.2) which is a well-known equation and is of great value in physics (where the symbol * denotes the complex conjugate).

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III. A FINITE-DIMENSIONAL HAMILTONIAN SYSTEM

Consider the spectral problem (1.1) and it's adjoint spectral problem

$$\psi_x = -M^T(u,\xi\psi), \qquad \psi = (\psi_1,\psi_2,\psi_3)^T,$$ * MERGEFORMAT (3.1)

A direct calculation shows that

$$\int_\Omega \psi^T \dot{M}\phi dx = 0$$

where $\dot{M} = \frac{d}{d\epsilon}|_{\epsilon=0}[M(u+\epsilon\delta u)]$. Then [12]

$$\Delta\xi = \left(\frac{\delta\xi}{\delta u_1},\frac{\delta\xi}{\delta v_1},\frac{\delta\xi}{\delta u_2},\frac{\delta\xi}{\delta v_2}\right)^T = \left(i\int_\Omega(\phi_1\psi_1-\phi_2\psi_2-\phi_3\psi_3)dx\right)^{-1}(\phi_2\psi_1,\phi_1\psi_2,\phi_3\psi_1,\phi_1\psi_3)^T$$ * MERGEFORMAT (3.2)

and

$$K\Delta\xi = \xi J\Delta\xi,$$ * MERGEFORMAT (3.3)

Now, suppose $\xi_k, k=1,2,\cdots,N$ is an eigenvalue of (1.1) and (3.1), $(\phi_{1k},\phi_{2k},\phi_{3k})^T$, $(\psi_{1k},\psi_{2k},\psi_{3k})^T$ are the eigenfunctions for ξ_k, $\Lambda = diag(\xi_1,\xi_2,\cdots,\xi_N)$. Then, the spectral problem (1.1) and it's adjoint spectral problem (3.1) can be rewritten as follows:

$$\begin{pmatrix}\phi_{1k}\\ \phi_{2k}\\ \phi_{3k}\end{pmatrix}_x = \begin{pmatrix} i\xi_k & u_1 & u_2\\ v_1 & -i\xi_k & 0\\ v_2 & 0 & -i\xi_k\end{pmatrix}\begin{pmatrix}\phi_{1k}\\ \phi_{2k}\\ \phi_{3k}\end{pmatrix} = M(u,\xi_k)\begin{pmatrix}\phi_{1k}\\ \phi_{2k}\\ \phi_{3k}\end{pmatrix},$$ * MERGEFORMAT (3.4)

$$\begin{pmatrix}\psi_{1k}\\ \psi_{2k}\\ \psi_{3k}\end{pmatrix}_x = -M^T(u,\xi_k)\begin{pmatrix}\psi_{1k}\\ \psi_{2k}\\ \psi_{3k}\end{pmatrix}.$$ * MERGEFORMAT (3.5)

Set $\Phi_j = (\phi_{j1},\phi_{j2},\cdots,\phi_{jN})^T, \Psi_j = (\psi_{j1},\psi_{j2},\cdots,\psi_{jN})^T, j=1,2,3$. We consider the following constraint:

$$u = (u_1,v_1,u_2,v_2)^T = (i\langle\Phi_1,\Psi_2\rangle, i\langle\Phi_2,\Psi_1\rangle, i\langle\Phi_1,\Psi_3\rangle, i\langle\Phi_3,\Psi_1\rangle)^T$$ * MERGEFORMAT (3.6)

Substituting (3.6) into(3.4),(3.5), we can get the Hamiltonian function F

$$F = f + f^*$$ * MERGEFORMAT (3.7)

where

$$f = i\langle\Lambda\Phi_1,\Psi_1\rangle - i\langle\Lambda\Phi_2,\Psi_2\rangle - i\langle\Lambda\Phi_3,\Psi_3\rangle + i\langle\Lambda\Phi_1,\Psi_2\rangle\langle\Lambda\Phi_2,\Psi_1\rangle + i\langle\Lambda\Phi_1,\Psi_3\rangle\langle\Lambda\Phi_3,\Psi_1\rangle$$

If the coordinates are as follows:

$$\begin{cases} y_1 = (\Phi_1,\Phi_1^{*T}), \ \ y_2 = (\Phi_2,\Phi_2^*)^T, \ \ y_3 = (\Phi_3,\Phi_3^*)^T,\\ z_1 = (\Psi_1,\Psi_1^*)^T, \ \ z_2 = (\Psi_2,\Psi_2^*)^T, \ \ z_3 = (\Psi_3,\Psi_3^*)^T.\end{cases}$$ * MERGEFORMAT (3.8)

The following theorem immediately holds.

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Theorem 3.1. On the constraint (3.6), (3.4) and (3.5) with their conjugate representations are equal to the Hamiltonian canonical system

$$\begin{cases} y_{jx} = \dfrac{\partial F}{\partial z_j}, \\ z_{jx} = -\dfrac{\partial F}{\partial y_j}, \end{cases} \quad j = 1,2,3. \qquad \text{* MERGEFORMAT (3.9)}$$

Define the Hamiltonian function as follows:

$$F_0 = f_0 + f_0^*, \quad F_1 = f_1 + f_1^*, \quad F_m = f_m + f_m^*, \; m = 2,3,... \qquad \text{* MERGEFORMAT (3.10)}$$

Where

$$f_0 = 2i\langle \Phi_1, \Psi_1 \rangle - 2i\langle \Phi_2, \Psi_2 \rangle - 2i\langle \Phi_3, \Psi_3 \rangle,$$

$$f_1 = 2i\langle \Lambda\Phi_1, \Psi_1 \rangle - 2i\langle \Lambda\Phi_2, \Psi_2 \rangle - 2i\langle \Lambda\Phi_3, \Psi_3 \rangle + 2i\langle \Phi_1, \Psi_2 \rangle\langle \Phi_2, \Psi_1 \rangle + 2i\langle \Phi_1, \Psi_3 \rangle\langle \Phi_3, \Psi_1 \rangle$$

$$f_m = 2i\langle \Lambda^m\Phi_1, \Psi_1 \rangle - 2i\langle \Lambda^m\Phi_2, \Psi_2 \rangle - 2i\langle \Lambda^m\Phi_3, \Psi_3 \rangle$$

$$+\sum_{j=1}^{m} 2i(\langle \Lambda^{j-1}\Phi_1, \Psi_2 \rangle\langle \Lambda^{m-j}\Phi_2, \Psi_1 \rangle + \langle \Lambda^{j-1}\Phi_1, \Psi_3 \rangle\langle \Lambda^{m-j}\Phi_3, \Psi_1 \rangle)$$

$$+\sum_{j=2}^{m} i(\langle \Lambda^{j-1}\Phi_1, \Psi_1 \rangle\langle \Lambda^{m-j}\Phi_1, \Psi_1 \rangle + \langle \Lambda^{j-1}\Phi_2, \Psi_2 \rangle\langle \Lambda^{m-j}\Phi_2, \Psi_2 \rangle$$

$$+\langle \Lambda^{j-1}\Phi_3, \Psi_3 \rangle\langle \Lambda^{m-j}\Phi_3, \Psi_3 \rangle + 2\langle \Lambda^{j-1}\Phi_2, \Psi_3 \rangle\langle \Lambda^{m-j}\Phi_3, \Psi_2 \rangle).$$

Theorem 3.2. On the constraint (3.6), the auxiliary problem of the spectal problem (1.1) can be written as the Hamiltonian canonical system

$$\begin{cases} y_{j,t_m} = \dfrac{\partial F_m}{\partial z_j}, \\ z_{j,t_m} = -\dfrac{\partial F_m}{\partial y_j}, \end{cases} \quad m = 0,1,2,..., \; j = 1,2,3. \qquad \text{* MERGEFORMAT (3.11)}$$

Thereom 3.3. [13,14] Suppose $(y_1, y_2, y_3, z_1, z_2, z_3)$ is an involutive solution of the Hamiltonian canonical equation systems (3.9) (3.11), then

$$\begin{cases} u_1 = i\langle \Phi_1, \Psi_2 \rangle, \\ v_1 = i\langle \Phi_2, \Psi 1 \rangle, \\ u_2 = i\langle \Phi_1, \Psi_3 \rangle, \\ v_2 = i\langle \Phi_3, \Psi_1 \rangle \end{cases}$$

satisfies the evolution equation (2.3).

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IV. THE COMPLETELY INTEGRABILITY OF THE HAMILTONIAN SYSTEM

Let [15]

$$\begin{cases} E_j^{(1)} = 2i\Phi_{1j}\Psi_{1j} - 2i\Phi_{2j}\Psi_{2j} - 2i\Phi_{3j}\Psi_{3j} + 2i(\Gamma_j^{(1,2)} + \Gamma_j^{(2,3)} + \Gamma_j^{(3,1)}) \\ E_j^{(2)} = \sum_{l=1}^{3} \Phi_{1j}\Psi_{1j} \\ E_j^{(3)} = 2i\Gamma_j = 2i \sum_{k=1,k\neq j}^{N} \frac{1}{\xi_j - \xi_k} (\mathrm{E}_k^{(2)}\mathrm{E}_j^{(2)}) \end{cases}$$ * MERGEFORMAT (4.1)

Thereom 4.1. The systems (3.9), (3.19) are the completely integrable systems in the Liouville sense. i.e.

$$\{F, E_j^{(l)}\} = 0, \quad l = 1,2; \quad j = 1,2,\cdots,N,$$

$$\{H_m, E_j^{(l)}\} = 0, \quad l = 1,2; \quad j = 1,2,\cdots,N,$$

$$\{H_m, H_n\} = 0, \quad m,n = 0,1,2,\cdots$$

$$\{F, H_m\} = 0, \quad m = 0,1,2,\cdots$$

By Arnold theorem [16], the systems (3.9) (3.19) are all the completely integrable systems in the Liouville sense.

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